Langevin models of turbulence: Renormalization group, distant interaction algorithms or rapid distortion theory?

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A new dynamical turbulence model is validated by comparisons of its numerical simulations with fully resolved, direct numerical simulations (DNS) of the Navier–Stokes equations in three-dimensional, isotropic, homogeneous conditions. In this model the small-scale velocities are computed using a Langevin, linear, inhomogeneous, stochastic equation that is derived from a quasi-linear approximation of the Navier–Stokes equations, in the spirit of rapid distortion theory (RDT). The values of the turbulent viscosity involved in our Langevin model are compared with a theoretical prescription based on the renormalization group and the distant interaction algorithms (DSTA) model. We show that the empirical turbulent viscosities derived from simulations of the Langevin model are in good quantitative agreement with the DSTA predictions. Finally, Langevin simulations are compared with DNS and large eddy simulations based on the eddy-damped quasi-normal Markovian closure. The Langevin RDT model is able to reproduce the correct spectrum shape, intermittency statistics, and coherent flow structures for both the resolved and the largest sub-grid scales. It also predicts the evolution of the resolved scales better than the alternative models. © 2003 American Institute of Physics. [DOI: 10.1063/1.1564826]

I. INTRODUCTION

The quadratic nonlinearity in Navier–Stokes equations is responsible for most of the difficulties encountered in describing and understanding turbulence. For example, it is responsible for the well-known closure problem in statistical theories of turbulence. In some respect, it is also responsible for our current inability to simulate flows at large Reynolds numbers. Indeed, in the Fourier-space representation, the nonlinear term of the Navier–Stokes equations involves interactions among three wave numbers. Among these we may focus on two types of triad interactions: Some include only wave numbers of comparable modulus (which we refer to as local in scale) and others involve two large and one small wave number (nonlocal). In the physical-space representation these two types of interactions correspond to two types of physical processes. The local interactions, involving eddies of comparable size, are the foundation of the Kolmogorov–Richardson picture of turbulence, by which energy is transferred from large to small scale via a scale-invariant cascade. The nonlocal interactions effect, for example, the advection and stretching of the small eddies by the large eddies. There are now good indications that the energy transfer is mostly due to local interactions, while the evolution of the small scales is principally governed by their interaction with the largest, more energetic scales. These observations led recently to a new interpretation of the statistical properties of turbulence where the local interactions are responsible for the $k^{-5/3}$ energy spectrum, while the nonlocal interactions are responsible for the deviations from Gaussianity observed at increasingly smaller scales (the so-called intermittency phenomenon).

When it comes to simulating the Navier–Stokes equations, the nonlocal interactions couple the largest scale of the flow with the smallest, dissipative scales. At large Reynolds numbers the range of scale to consider is thus enormous and the computational resources become insufficient.

An interesting way to overcome these difficulties and create a model with notable numerical potential would be to replace the Navier–Stokes equation by a simplest, linear equation, in which the quadratic nonlinearity is replaced by processes mimicking the action of local and nonlocal interactions. However, in numerical simulations, local and nonlocal interactions are easy to handle only in the presence of a spectral gap. In the sequel we will, therefore, distinguish between “RDT interaction” that characterizes the interaction between resolved and sub-grid scales and “nonlinear self-interaction” that characterizes the interactions between sub-grid scales. Due to the absence of spectral gap, these interactions include both local and nonlocal interactions. Specifically, we shall focus on the model obtained when the Navier–Stokes equations in Fourier space are replaced by a linear stochastic equation of the Langevin type:

$$(\partial_t + u k^2 + v_i(k,t)k^2)u_i(k,t) = f_i(k,t) + F_i,$$

where $v$ is the molecular viscosity; $F_i$ the external (usually
large-scale) forcing; \( u_i(k,t) \) is a dynamical viscosity; and \( f_i \) is a random, solenoidal, dynamical forcing. Thus \( u_i \) and \( f_i \) represent linear replacements of the nonlinear advection term. In the standard phenomenology, \( u_i \) is assumed to represent interactions of wave number \( k \) with wave numbers \( > k \) and \( f_i \) representing interactions with wave numbers \( < k \). Different models result from different prescriptions for the dynamical viscosity and forcing. The prescriptions themselves are based on an approximation of the Navier–Stokes equations in order to restore energy conservation. In effect this leads to a Navier–Stokes equation with more or less theoretical soundness. In this paper we mainly focus on practical aspects by selecting models which have met some success in providing tools for the simulation of Navier–Stokes equations at large Reynolds number.

One of the first models along these lines is based on a direct interaction approximation (DIA).\(^7\) In this approximation the phase correlation embodied in the nonlinear term of Navier–Stokes equations is replaced by a damping term in order to restore energy conservation. In effect this leads to a Langevin equation in which the random forcing (i.e., interactions between inertial and energy containing range wave numbers) is modeled by modified nonlinear terms. Other models like the eddy-damped, quasi-normal Markovian model (EDQNM)\(^8,9\) or the test field model (TFM)\(^10\) are built upon the DIA by introducing white-noise statistics in the nonlinear term (i.e., in the forcing). Statistical structure is assumed to be white in time and Gaussian in space. The remaining ingredient of an even more radical step towards that direction is taken in the so-called renormalization-group approach (RG) (McComb\(^11\)). In this approach, pioneered by Foster et al.,\(^12\) the forcing \( f \) is externally prescribed, and its statistical structure is assumed to be white in time and Gaussian in space. The remaining ingredient of the Langevin model (the viscosity) is then derived through recursive wave number band elimination via a first-order expansion in the nonlinearity strength. Fournier and Frisch\(^13\) investigated the relationship between the Foster et al. theory and the EDQNM closure model. Later, Yakhot et al.\(^14\) and others such as Canuto\(^15\) have applied the RG approach to practical turbulence problems by using an additional hypothesis which lacks theoretical grounds.\(^16\)

Another elimination scheme to compute the turbulent viscosity is proposed by Kraichnan\(^8\) using the distant interaction approximation (DSTA). The essence of the DSTA is that the dynamics of modes in the inertial and dissipation ranges are assumed to be dominated by interactions more or less local in wave number that are modeled by white noise force acting against a dynamical viscosity, and this effective viscosity is estimated by extrapolation from the small contributions of interactions that are very nonlocal in wave number.

These Langevin models have had some successes and weaknesses as closure models of turbulence. The most prominent flaw in the Langevin model based on DIA is its failure to give the Kolmogorov value of the energy-spectrum in the inertial range. The explanation from this failure is provided by Kraichnan\(^17\) who shows that the DIA overestimates the effect of large scale convection of the inertial ranges. Models based on the prescribed white in time Gaussian forcing perform more satisfactorily in this respect. For example, Canuto and collaborators\(^15,18,19\) report satisfactorily modeling of spectra and Reynolds stress in isotropic or stratified turbulence. The turbulent viscosity prescriptions of RG and EDQNM are also used in large eddy simulations (LES) as models for the effect of the unresolved small scales on the resolved ones (e.g., Ref. 9). In this special case of LES, the turbulent forcing is in some sense exactly modeled, since the nonlinear interactions between large scales are exactly taken into account. Failures of the Langevin models occur mainly when considering the small-scale behavior. On the one hand, exact analytical or easy numerical computations are possible only using a simple prescription for the turbulent forcing. This motivates the usual choice for \( f \) of a noise that is white in time and Gaussian in space. On the other hand, this simple choice leads to undesirable and unphysical features for the small scales; for example, such a noise leads to a phase scrambling which prevents the formation of coherent structures in the flow. In a Langevin equation forced by Gaussian white noise, one can observe development of a \( k^{-5/3} \) energy spectrum without formation of small scale vorticity filaments.\(^20\) Also, an additive Gaussian forcing necessarily induces Gaussian statistics for the velocity field. In homogeneous turbulence these statistics are only observed at the largest scales. At small scales the statistics become increasingly non-Gaussian, with broad, stretched-exponential tails in the probability distribution functions (PDFs).

These difficulties could be circumvented by the introduction of a more realistic turbulent forcing built from the Navier–Stokes equations. At the moment, however, there is no systematic theory of 3D turbulence leading to an analytical prescription for the forcing. In the present paper, we show how empirical splitting of the dynamics into large and small scales and a quasi-linear approximation at small scales result in a new Langevin model of turbulence where the turbulent forcing is numerically computed. This model can be used to perform high Reynolds-number simulations at low computational cost. It has been previously numerically validated in two-dimensional (2D) by comparison with DNS\(^21\) and used to perform fast numerical simulations of 2D turbulence.\(^22\) In the 3D case a preliminary analysis of the interactions in a turbulent boundary layer at very large Reynolds number\(^23\) indicates that the RDT interaction prevail over nonlinear self-interaction at small scales, a situation reminiscent of 2D turbulence. In this paper we complete this analysis and the validation of the dynamical turbulent model using 3D numerical simulations similar in spirit to those performed for 2D.

II. THE RDT MODEL

The dynamical turbulent model is obtained from the forced Navier–Stokes equations in physical space

\[
\partial_t u + u \cdot \nabla u = -\nabla p + v \Delta u + F,
\]

where \( u \) is the velocity, \( p \) is the pressure, \( v \) is the molecular viscosity, and \( F \) is the external forcing. The forcing can be provided by boundary conditions or externally fixed, e.g., keeping constant a designated small-wave number Fourier
mode coefficient in a numerical simulation. Below we consider both forced and decaying cases (i.e., with $\mathbf{F}=0$). To derive the approximate model, we proceed as in Ref. 21 and introduce a filter function $G(\mathbf{x})$ in order to separate the large and small scales of the flow. In our numerical procedure the filter $G$ will be taken as a wave number cut-off function. In this case the large scales (noted with an over-bar) will be identified with a wave number smaller than the cut-off value $k_c$ and the small scales (noted with a prime) with wave numbers larger than $k_c$. This identification of scales and wave numbers is not straightforward for any filter especially filters defined in the physical space, but because we chose a cut-off filter, the identification will be used later on. We checked that the results are insensitive to the choice of the filter, provided the Fourier transform of the latter decays fast enough as $k \to \infty$. Thus, we decompose the velocity field into resolved (large-scale) and sub-grid (small-scale) components that are defined in the physical space from the inverse Fourier transform of their corresponding Fourier components

$$u(\mathbf{x},t) = \bar{u}(\mathbf{x},t) + u^e(\mathbf{x},t).$$

(3)

We now obtain separate equations for the resolved and sub-grid scales of motion. Equations for the resolved scales of motion are obtained by application of the filter [Eq. (3)] to the individual terms in (2). The result is

$$\partial_t \bar{u}_i + \partial_j \bar{u}_i \bar{u}_j + \partial_j (\bar{u}_i u'_j + \bar{u}_j u'_i) + \partial_j u'_i u'_j = - \partial_i P + v \Delta \bar{u}_i + F_i, \quad \partial_j \bar{u}_j = 0.\quad (4)$$

The large-scale motions depend on the small-scale motions through two sub-grid stress terms, the sub-grid–sub-grid product $\bar{u}_i u'_j$ and the resolved/sub-grid cross term $\bar{u}_i u'_j + u'_i u'_j$. These terms can be computed via integration of the sub-grid component of (2)

$$\partial_j u'_i + \partial_j (u'_i u'_j - u'_i u'_j) = - \partial_i P + v \Delta u'_i, \quad \partial_j u'_j = 0.\quad (5)$$

Several terms contribute to the nonlinear evolution: The term \textit{linear} in small scales, involving the product of a resolved scale and a sub-grid component, and the term \textit{nonlinear} in small scales, involving two sub-grid components. The essence of the turbulent model is to keep in (5) the \textit{linear terms} as they are, while replacing the \textit{nonlinear term} by an effective term. In the spirit of previous Langevin models, we shall split this effective term into a stochastic forcing (representing interaction of the sub-grid velocity with modes near the cut-off scale) and a turbulent viscosity (representing the interaction with the very smallest sub-grid scales of the flow). This procedure can be expected to give good results only if the terms \textit{nonlinear} in small scales—which contains all of the approximation—are less important in the dynamics than the term involving large scales. The model we propose is not exactly an LES model because it requires the computation of an approximate equation for small scales (or unresolved scale in real LES model). Consequently, the computational cost of such a model will be higher and will depend of the computational cost of the equation for small scales. In the case of 2D turbulence, this model was used to perform fast numerical simulations using a semi-Lagrangian method exploiting the linearity of the small-scale equation, resulting in a gain in computational time up to 100 compared to DNS. In 3D we expect a similar method to lead to at least the same gain. As it will be shown later, the stochastic forcing, generally numerically costly, does not appear as crucial as turbulent viscosity. Therefore, one saves significant computational resources by using only turbulent viscosity. It is very difficult to give an estimation of the real gain because it is dependent on the error we can tolerate compared to an equivalent DNS at the same Reynolds number. For small Reynolds numbers, the approximated model is similar, in spirit, to the estimated models where unresolved scales are estimated from the resolved scales.\textsuperscript{24,25}

\section*{III. A PRIORI VALIDATION}

To check the quasi-linear hypothesis, we may first estimate the order of magnitude of each term separately and use \textit{a priori} tests on a velocity field obtained from a DNS of the Navier–Stokes equations. This numerical integration and all subsequent ones are performed using a standard, pseudo-spectral code with de-aliasing, described in Ref. 26. The spectral equation simulated is

$$(\partial_t + v k^2) \mathbf{u}(\mathbf{k},t) = [[\mathbf{u},\mathbf{u}])(\mathbf{k},t) + \mathbf{F},$$

where $[[\mathbf{u},\mathbf{u}]]$ is the nonlinear term defined by

$$[[\mathbf{u},\mathbf{v}]]_m(\mathbf{k},t) = -i P_{mj} k_t \sum_{p+q=-k} u_j(\mathbf{p},t) v_j(\mathbf{q},t),$$

and $P_{mj}$ is the projector defined by $P_{mj} = \delta_{mj} - k_m k_j / \mathbf{k}^2$ and the summation is implied by repeated indices. We use 256 discretization in all our simulations (Table I), reaching Reynolds numbers based on the Taylor microscale ranging from 80 to 200. The \textit{a priori} tests are performed on two velocity fields resulting from (i) a forced simulation at $R_{\lambda} = 200$ (FDNS) where the forcing is produced by maintaining constant all the modes with a module $k<1.5$ and (ii) a decaying simulation (D-DNS) with $R_{\lambda} = 80$–57. In order to reach high enough Reynolds number while maintaining a wide enough scale separation, the two DNS simulations are slightly under-resolved. The ratio of the maximum wave number $k_{max}$ to the Kolmogorov wave numbers $k_d$ are, respectively, $k_{max}/k_d = 0.72$ for D-DNS, and $k_{max}/k_d = 0.84$ for F-DNS. The initial analysis time ($t=0$) for these two simulations is chosen after several turnover times. (All the simulations are initialized with a random vorticity field at large scales.) From the velocity fields at $t=0$, we compute the various nonlinear terms appearing in Eqs. (4) and (5), for different choices of the cut-off wave number $k_c$

$$[[\mathbf{u},\mathbf{u}])(\mathbf{k},t) = [[[\mathbf{u},\mathbf{u}])(\mathbf{k},t) + [[[\mathbf{u},\mathbf{u}]')(\mathbf{k},t)$$

$$+ [[[\mathbf{u},\mathbf{u}]')(\mathbf{k},t) + [[[\mathbf{u},\mathbf{u}]')(\mathbf{k},t).$$

(8)

The $k_c$ are chosen large enough in order to keep the greatest part of the total energy in the largest scales. Each of these nonlinear terms will be separated into their large scale components $([[\mathbf{u},\mathbf{u}]])$ and their small scale component $([[\mathbf{u},\mathbf{u}]'])$. A summary of our results is shown in Fig. 1 for the decaying case and in Figs. 2 and 3 for the forced case. At large scales one sees that the contribution from the nonlinear, self-interactions (involving two resolved quantities) domi-
nates the other contributions by several order of magnitudes. The exact value of the difference depends on the value of \( k_c \), but it remains large for any \( k_c \) in the inertial range. At small scales the situation changes, and the dynamics becomes dominated by RDT interactions, involving the resolved vorticity and the sub-grid velocity. This domination is less pronounced (by about only one order of magnitude) but extends throughout the scale range. The next-order term comes from the nonlinear self interactions, first between resolved scales, then between sub-grid scales. The ordering of terms we obtain in this 3D situation is qualitatively similar to that obtained in 2D turbulence.21 The main difference is the relative difference between terms, which is much larger in the 2D situation. This is probably due to the steeper energy spectrum of 2D turbulence that, for a given cut-off, allows a clearer separation between resolved and sub-grid scales; since the latter is much smaller than the former, the ratio of the quantities is smaller (see Fig. 4). Nevertheless, these \textit{a priori} tests show that the RDT interactions dominate the dynamics at small scale in 3D. This motivates our constructing a model where they are taken exactly into account.

A similar \textit{a priori} analysis has been conducted by Carlier \textit{et al.},23 on experimental results of a shear-flow boundary layer at high Reynolds number \( R_A \) up to 567 at different distances from the wall. In these tested situations the second-order product involving small-scale velocities is at least one order of magnitude smaller than the main contribution at small scales. The same numerical analysis has also been conducted for shear flows and a planetary boundary layer27,28 at the highest Reynolds numbers available with DNS. All of these results corroborate our \textit{a priori} analysis of 3D homogeneous turbulence at moderate Reynolds number.

\section*{IV. THE RDT LANGEVIN MODEL}

The \textit{a priori} analysis shows that the dynamics at small scales is dominated by the RDT term involving interactions with the large scales. These terms involve one additive part

<table>
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<th>Simulations</th>
<th>L.S. forcing</th>
<th>RDT Resolution</th>
<th>Cut-off ((k_c))</th>
<th>Dissipation</th>
<th>Turb. viscosity</th>
<th>Turb. forcing</th>
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<td>No</td>
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<td>1.4 \times 10^{-3} \Delta</td>
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<td>...</td>
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**TABLE I. Description of the simulations.**

\[ \text{FIG. 1. Comparison of the spectrum of each part of the nonlinear term in the Navier–Stokes equations after a scale separation (3).} \]

\[ \text{FIG. 2. The same test as in Fig. 1 for a case of forced turbulence (F-DNS) with } k_c = 8 \text{ and } k_c = 16. \]
(the $[\langle u' \bar{u} \rangle''']$ term) and a multiplicative part (terms like $[[u', u']]$). These RDT interactions can be seen as a part of the turbulent forcing arising in Langevin models of turbulence. However, contrary to other Langevin models this forcing is not postulated a priori but computed using the large-scale equation. Because this equation is coupled to the small-scale equation via the Reynolds stresses, the resulting forcing is non-Gaussian and exhibits finite-time correlation (Ref. 4 has a detailed analysis).

In 2D we checked that retaining this forcing is alone sufficient to explain the dynamical properties of the small scales. In 3D, where strong vortex stretching appears, one may suspect that other types of interactions may play a non-negligible role for the small scales. To check this point we conduct several dynamical tests by comparing the results of a DNS of the Navier–Stokes equations with simulations in which nonlinear terms at small scales are either removed (as in the 2D RDT model) or replaced by a turbulent viscosity and a stochastic forcing, to take into account the interactions with scales near $k_c^{-1}$ or very much smaller. The equation for the small wave numbers ($k<k_c$) is obtained by taking the filtered part of Eq. (6)

$$\partial_t \bar{u}(k,t) + [[u,u]](k,t) = -vk^2 \bar{u}(k,t) + \bar{F}.$$  (9)

The equation for large wave numbers without approximation is deduced by subtracting Eq. (9) from Eq. (6)

$$\partial_t u'(k,t) + [[u,u]]'(k,t) = -vk^2 u'(k,t).$$  (10)

Then, the sub-grid part ($k>k_c$) of the interactions between large wave numbers components ([[$u', u'']]'(k,t)) is removed from Eq. (10) and replaced by a stochastic forcing ($f(k,t)$) and turbulent viscosity ($v \nu(k^2 u'(k,t))$). The couple of approximated equations then becomes

$$\partial_t \bar{u}(k,t) + [[u,u]](k,t) = -vk^2 \bar{u}(k,t) + \bar{F},$$  (11)

$$\partial_t u'(k,t) + [[u,u]]'(k,t) = -[[u', u']]'(k,t) - (v + \nu(k^2)) k^2 u'(k,t) + f(k,t).$$  (12)

Equation (12) is in fact a generalization of the rapid distortion theory, RDT$^{29}$ where turbulent viscosity and forcing terms have been added. This approach is computationally rather expensive: Splitting the equations of motions between resolved and sub-grid component leads to additional Fourier transforms and increases the computational time by a factor of 3. This sets a practical limit on the tests we could perform with our workstations. Therefore, we conduct most of the simulations only for approximately 2.5 turnover times, and we checked the time dependence of the results with two longer simulations (D-DNS and D-RDT24-V1). Due to the number of simulations required in this analysis, we restrict ourselves to a 256$^3$ grid, rather than expanding the validation studies with other simulations at higher Reynolds number.

**A. Tests with a conservative RDT model**

We first test the RDT model without turbulent viscosity and forcing. Comparisons are made for several quantities, often used as diagnostics or forecasters in 3D turbulence. The comparisons of spectra for both the forced case and the decaying case are shown in Figs. 5 and 6, respectively. In
both cases the conservative RDT solution leads to an energy spectrum which is a little bit shallower than the DNS spectrum near the cutoff and becomes steeper in the far dissipative range. This effect is more pronounced in the decaying case due to the flatter spectrum at large scale.\textsuperscript{4} This difference between the DNS and the RDT model shows that the effect of the RDT approximation leads to a reduction in the rate of the energy transfer at scales smaller than $k_c^{-1}$. This effect can also be seen in the plot of energy transfer (Fig. 7). In this plot we compare the energy transfer $T(k) = (\hat{\alpha}_k + 2\nu k^2)E(k)$ from the initial field of the simulations D-DNS (line) and D-RDT24 (cross). One can see an overestimation of $T(k)$ just behind the cutoff which corresponds to an underestimation of the energy cascade. This effect is even more visible in the energy fluxes $\Pi(k) = \sum_{k > q} T(q)$ shown in Fig. 8. This analysis clearly points out the importance of small-scale effects that may act partly as a turbulent viscosity and thereby prevent the accumulation of energy near the cut-off scale. The question is how to model these effects. We try two strategies: First, an empirical model explicitly using the neglected nonlinear sub-grid term, and second, a theoretical model based on analytical computations using the DSTA or RG method. From a computational viewpoint the second strategy is more satisfying since it leads to a parameter-free model of turbulence, albeit of uncertain validity. From a practical point of view, we also pursue the first strategy, devoid of theoretical simplifying assumptions (like scale separation or smallness of a parameter) to help clarify the requirements for a nonconservative RDT model.

\section*{B. Empirical turbulent viscosity and forcing}

\subsection*{1. Energy transfer}

The accumulation of energy near the cut-off scale can, in part, be attributed to RDT interactions at small scales that were discarded with the nonlinear, sub-grid product. We can first try to enhance the dissipation at this scale. We run a RDT simulation with first only a turbulent viscosity. The simplest choice is to take an isotropic, positive, turbulent viscosity with a wave number dependence related to the spectrum of the neglected nonlinear, sub-grid terms neglected. The formula used is

$$v_1(k)k^2|\mathbf{u}^t(k)| = C_1[|\mathbf{u}^t(k)|^4](k),$$

where $|\cdot|(k)$ denotes the modulus averaged over the spherical shell $k$. Adjusting the constant $C_1$ to 0.025, we can guarantee the equivalence of the energy decay between the direct simulation D-DNS and the approximated simulation D-RDT24-V1 (Fig. 9).

In order to check the evolution of the error of the total energy, the simulations D-DNS and D-RDT24-V1 are extended up to $t = 1.8$, corresponding to 7 turnover times. The inset of Fig. 9 indicates that the error is stabilized after $t = 1$. The spectrum of the turbulent viscosity for the simulation in the decaying case (D-RDT24-V1) is shown Fig. 10. The prescription given by Eq. (13) leads to an almost constant turbulent viscosity in $k$. This correction leads to a very good correction of the RDT spectrum in the inertial range that now is almost indistinguishable from the DNS spectrum (Fig. 6). However, at smaller scales this procedure steepens the RDT spectrum thereby increasing the difference between
the RDT model and the DNS. This effect is due to the fact that the prescription gives a nonzero turbulent viscosity at very small scales.

A more sophisticated prescription of the turbulent viscosity is to use only that part of the nonlinear self-interactions among small scales that is correlated with the velocity field. This procedure makes sense insofar as, by definition, the turbulent viscosity will act as a positive multiplicative coefficient on the velocity field itself. Therefore, we try to figure out what is the evolution in space of the correlation between the velocity field and the nonlinear subgrid terms. The result of this correlation computed from the initial velocity field at \( t = 0 \) is given in Fig. 11. The correlation is positive for scales up to approximately half the cut-off scale, and negative at smaller scales. We define a new prescription of the turbulent viscosity based on this behavior.

Keeping an isotropic turbulent viscosity it seems reasonable to use the following wave number dependence:

\[
v_2(k) = H(k) \frac{|\mathcal{F}(\mathbf{u}', \mathbf{u}')|^2(k)}{2k^2E(k)},
\]

where \( E(k) \) is the energy spectrum, and \( H(k) \) is a low pass filter which is equal to 1 for \( k < 2k_c \).

Since the part of the neglected term correlated with the velocity can be seen as a turbulent viscosity, we tried to treat the noncorrelated part as a stochastic forcing \( f(k) \). We use a turbulent forcing that is \( \delta \)-correlated in time with Gaussian statistics, and a wave number dependence defined by subtracting the dissipation from the turbulent viscosity from the total variance of the neglected nonlinear term. The resulting formula for the stochastic forcing is

\[
\langle f_i(k, t) f_j(k', t') \rangle = \mathcal{F}(k) P_{ij}(k) \delta(k + k') \delta(t - t'),
\]

where \( P_{ij} \) is the projector defined in (7), and the spectrum \( \mathcal{F}(k) \) is given by

\[
\mathcal{F}(k) = ||[\mathbf{u}' \cdot \mathbf{u}']||^2(k) - 2v_2k^2E(k).
\]

The shape of the turbulent viscosity \( v_2(k) \) and the spectrum \( \mathcal{F}(k) \) of stochastic forcing are shown Fig. 12 at two different times from the simulation of decaying turbulence D-RDT24-F-V2. It is remarkable that the shape of this new

\[
\text{FIG. 9. Comparison of the decay of the energy between the direct simulation D-DNS and the approximated simulations D-RDT24 and D-RDT24-V1. The percentage errors (} \varepsilon = 100 \times (E_{\text{RDT}} - E_{\text{DNS}})/E_{\text{DNS}}) \text{ are plotted in the inset.}
\]

\[
\text{FIG. 10. Spectrum of the turbulent viscosity } v_{1i}(k) \text{ computed by Eq. (13) in the simulation D-RDT24-V1.}
\]

\[
\text{FIG. 11. Spectrum of the covariance between the velocity } \mathbf{u}' \text{ and the nonlinear term at small scales } [\mathbf{u}' \cdot \mathbf{u}'] \text{ computed from the simulation D-RDT24-F-V2 with } k_c = 24.
\]

\[
\text{FIG. 12. Spectra } \mathcal{F}(k) \text{ of the stochastic forcing } f(k) \text{ (symbols) and the turbulent viscosity } v_2(k) \text{ (line) computed respectively, by Eqs. (16) and (14) for the simulation D-RDT24-F-V2 at two different times: } t = 0 \text{ (solid line and circle) and } t = 0.48 \text{ (dashed line and cross).}
\]
definition of turbulent viscosity is similar to the shape with the previous definition [Eq. (13)], where the constant \(C_1\) was adjusted to obtain good agreement of the approximated simulation with the equivalent DNS.

2. Spectra and structures

We perform three simulations of decaying turbulence with this new definition for \(v_2(t)\) and \(F(k)\): one with only a turbulent viscosity (D-RDT24-V2), one with only the stochastic forcing (D-RDT24-F), and the third with both corrections (D-RDT24-V2-F). The comparison of the energy spectra for the two previous simulations and the corresponding DNS is shown Fig. 13. Each of the corrections to the RDT simulations seems to give good results, although the simulation with only the turbulent viscosity seems to give a somewhat more accurate result. It is interesting to notice that the two corrections taken separately are effective in dissipating the accumulation of energy near the cut-off scale. Nevertheless, we perform a last simulation (D-RDT24-V2-F) with both turbulent viscosity and stochastic forcing. The comparison of the energy spectrum with the equivalent DNS is given in Fig. 14. The result for the energy spectrum is very similar to the results of the simulation with only the turbulent viscosity. This means that the stochastic forcing we chose is not relevant in this case. The effect of the two additional corrections to the RDT simulations can be seen on the plots of the energy transfer (Fig. 7) and the energy fluxes (Fig. 8). We clearly see a reduction of the underestimation of the energy transfer for the scales that are just smaller than the cut-off scale, but the correction is too dissipative at very small scales. The comparison of the energy decay between the approximated simulation with the two corrections and the corresponding DNS is as good as the comparison with the previous definition of the turbulent viscosity \(v_1(t)\) where the multiplicative constant \(C_1\) was adjusted (Fig. 15). The maximum of the energy error is less than 1%.

In physical space the RDT model successfully reproduces the coherent structures observed in DNS (Fig. 16). One sees that, in the absence of corrections, the vorticity filaments in high vorticity regions are more intense than in the equivalent DNS. This undesirable feature is removed by the addition of the correction. This enhances the dissipation of these coherent structures, leading to filaments with intensity comparable with that of DNS (Fig. 16). The same results are obtained with the two simulations with only turbulent viscosity and only stochastic forcing. Thus, we can conclude from the success of the RDT model that the strength and distribution of coherent vortex filaments in 3D turbulence are largely governed by RDT interactions, coupling the largest scales and the smallest scales of the filaments.

3. PDFs

The statistical comparisons so far are only for low-order moments. A more stringent test of the quality of our model can be obtained from statistical quantities that measure the intermittent properties of the velocity field, hence the high-order correlations. A standard, scale-dependent, statistical indicator is based on the velocity increment over a distance \(\ell\), defined by

\[
\delta u_i = u(x + \ell) - u(x).
\]
When $\ell$ is in the direction of $u$, the increment is called longitudinal. When $\ell$ is perpendicular to $u$, the increment is transverse. The PDF of the longitudinal and transverse velocity increments at length $\ell = L/256$ are shown in Figs. 17 and 18 for the decaying case. In both cases the PDFs are computed using only the velocity field at the end of the simulation. One observes that the RDT with no additional correction displays much wider wings than the DNS, thereby indicating stronger intermittency. With the turbulent viscosity and stochastic forcing, the wings shrink, and the RDT PDFs become comparable to the DNS PDFs. The role of the nonlinear term in small-scales intermittency is investigated in detail in Ref. 4.

C. Theoretical turbulent viscosity and forcing

1. Definition

The main conclusion of our study using an empirical viscosity and forcing is that there is no need for an additional stochastic forcing in our model; apparently its contributions are well represented (at the resolutions we are able to reach) by the retained coupling with the large scales. Can the same
conclusion be reached with theoretical viscosity and forcing. We investigate this question using theoretical model based on renormalization group (RG)\textsuperscript{15,30} and the distant-interaction algorithm (DSTA).\textsuperscript{9} These methods provide an explicit expression for the turbulent viscosity in spectral space. In the inertial range, this expression amounts to
\begin{equation}
v_i(k) = C_i \frac{k}{k_i}^{-4/3} \sqrt{E(k_i)} / k_i.
\end{equation}
where \(k_i\) is a wave number in the inertial range, \(E\) the energy spectrum, and \(C_i\) is a constant depending on the model. The Wyld diagrammatic method of Canuto \textit{et al.}\textsuperscript{15} yields \(C_i = \sqrt{3/20} = 0.3873\). The renormalization group expansion of Giles\textsuperscript{30} yields \(C_i = \sqrt{37/12\pi^2} = 0.3153\), close to the estimate of Canuto. In the DSTA the constant depends on a parameter \(\beta\) characterizing the degree of nonlocality of the interaction. For the contribution at \(k\) of all modes with wave number greater than \(\beta k\), DSTA yields
\begin{equation}
C_i(\beta) = \sqrt{7/6} \beta^{-2/3} = 0.3416 \beta^{-2/3}.
\end{equation}
There is no such exact result for the turbulent forcing. The constraint of energy conservation links correlation of the forcing to the magnitude of the turbulent viscosity. However, the detailed statistics of \(f\) are under-constrained. A classical choice is to assume statistics for \(f\) which are white in time and Gaussian in space. The conservation constraint then imposes a relation between the stationary energy spectrum \(E(k)\), the turbulent viscosity, and the forcing spectrum \(F(k)\), viz.,
\begin{equation}
\mathcal{F}(k) = 2v_i(k)k^2 E(k).
\end{equation}
In the presence of a large-scale field, the Langevin equation takes the more general shape
\begin{equation}
(\partial_t + v k^2 + u_i(k,t)k^2) u_i(k,t) = A_{ij} u_j(t) + F_{ij}(k,t) \xi_j,
\end{equation}
where \(\xi_j\) is a Gaussian vector of unit variance [Eq. (12)]. This slightly changes the estimate in Ref. 31. Indeed, the general solution of (21) with initial condition \(u_0\) is
\begin{equation}
u(t) = e^{At} u_0 + \int_0^t e^{A(t-s)} F(s) \xi ds,
\end{equation}
where \(A\) is the general operator
\begin{equation}
A_{ij} = -(v k^2 + u_i(k,t)k^2) \delta_{ij} + A_{ij}.
\end{equation}
This solution is linearly dependent on the force. If the latter is Gaussian, the solution is also Gaussian, and its statistics are fully characterized by the second moments. Using the Gaussian forcing (Ref. 31 has details), the second moments can be expressed as
\begin{equation}
\langle u_i(t) u_j(t) \rangle = \langle F_{ij}^{\dagger} F_{jk} \rangle F_{kl},
\end{equation}
where \(B\) is the matrix
\begin{equation}
B = \int_0^t e^{A(t-s)} \delta(t-s) ds.
\end{equation}
From the definition of \(B\) it can be shown that, in the limit \(t \to \infty\), it reaches a stationary state and satisfies the Lyapunov equation
\begin{equation}
A^+ B^+ + B^+ A = -I.
\end{equation}
Finally, taking the trace of (24) and taking the limit \(t \to \infty\) to obtain a stationary state, we get
\begin{equation}
E(k) = \langle \text{Tr}(F^+ B^+ F) \rangle.
\end{equation}
Using \(\langle F^+ F \rangle = I \mathcal{F}(k)\), where \(\mathcal{F}\) is the force spectrum, and \(I\) the unit matrix, we finally obtain the condition for stationary energy spectrum as
\begin{equation}
\mathcal{F}(k) = D E(k) / \text{Tr}(B^+) .
\end{equation}
Here, \(D\) is the space dimension. Note that when \(A = -v k^2 I\), then \(B^+ = I/(2v k^2)\), and we recover (20).
We try both prescriptions and found that, unless the large scale eddy turnover time is comparable to the small scale eddy turnover time at the cutoff, the simple estimate (20) provides a reasonable stationary state.

\section{2. Results}

Figure 19 shows the results obtained with the RG model of Canuto.\textsuperscript{15} The energy spectrum indeed exhibits a clean \(k^{-5/3}\) shape over several decades in \(k\), as built into the model. Yet on Fig. 20, one can see that the structures look very different from the DNS and that some phase information has been lost through the Gaussian forcing. This impression is confirmed by inspection of the PDFs of velocity increments.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig19}
\caption{Energy spectrum of a RG simulation (F-RG) on a 128\textsuperscript{3} grid (line) and the RDT simulation F-RDT10-V3 (dashed line). (See Table I for definition of simulations.)}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig20}
\caption{Iso-contours of vorticity magnitude (2.2 \(u_{\text{rms}}\)) for (a) a RG simulation (F-RG) on a 128\textsuperscript{3} grid and (b) the RDT simulation F-RDT10-V3.}
\end{figure}
(Fig. 21), which are Gaussian at all scales and do not display the usual wide tails observed in the DNS at small scales.

We then try to include only a turbulent viscosity at small scales, presuming that the turbulent forcing can be satisfactorily taken into account through the coupling with the large scales. However, we find that the turbulent viscosity predicted by the RG method is too large to lead to a stationary state with a $k^{-5/3}$ energy spectrum. Such a state could only be reached upon lowering the constant $C_v$ by about an order of magnitude. Figures 19–21 show the energy spectrum, the filaments, and the velocity PDFs for the simulation F-RDT10-V3 in a $128^3$ grid with $C_v = 0.24$. This simulation provides a satisfactory comparison with the DNS. One can even compute the correlation coefficient between a DNS at relatively high Reynolds number and the model that is supposed to model a simulation at infinite Reynolds number.

The DNS is performed on a $256^3$ grid. The correlation coefficient is computed by

$$\text{Correlation (Model,DNS)} = \frac{\omega_{\text{Model}} \cdot \omega_{\text{DNS}}}{\sqrt{\left| \omega_{\text{Model}} \right|^2 + \left| \omega_{\text{DNS}} \right|^2}}.$$  (29)

The result of this comparison is shown Fig. 22. The two simulations are well correlated for a few turnover times occur.

We try to change the resolution and the cut-off $k_c$ to determine the dependence of the constant $C_v$ with respect to these quantities. Each determination is very time consuming, and we can handle only a few cases, summarized as follows. We perform three RDT simulations (F-RDT05-V3, F-RDT10-V3, and F-RDT15-V3) on a $128^3$ grid with three different cut-off wave numbers, respectively, $k_c = 5.5$, $k_c = 10.5$, and $k_c = 15.5$. The turbulent viscosity used for these

![Fig. 21. PDF of longitudinal velocity increments [Eq. (17)] for (a) a RG simulation (F-RG) on a $128^3$ grid and (b) the RDT simulation F-RDT10-V3. For each graph the six curves correspond to six different values of the increment $\ell$.]

![Fig. 22. Correlation coefficient [def. Eq. (29)] between the large scales ($k < 11$) of the simulation F-DNS and the equivalent scales of (a) the simulations F-RDT11 (line), (b) the simulations F-RDT11-V1 (dotted line), (c) the LES simulation F-LES on a $32^3$ grid with the spectral eddy viscosity model (Ref. 32) (dashed line), and (d) the RDT simulation F-RDT10-V3 with the turbulent viscosity given by Eq. (30) with $C_v = 0.24$ and $k_c = 10.5$ (dash-dot-dashed line). $\tau = 0.35$ is the turnover time.]

![Fig. 23. Energy spectra of the three RDT simulations (F-RDT05-V3, F-RDT10-V3, and F-RDT15-V3) on a $128^3$ grid with the turbulent viscosity given by Eq. (30) and for three different cut-off wave numbers: $k_c = 5.5$, $k_c = 10.5$, and $k_c = 15.5$ (the cut-off scales are represented by corresponding vertical lines). The corresponding values of the constant $C_v$ are, respectively, 0.16, 0.24, and 0.35.]

where $E$ is the energy spectrum close to a Kolmogorov spectrum $\sim k^{-5/3}$. There seems to be a weak but discernible dependence of $\nu_c$ on $k_m$, which in our case is $k_m = 41$. The second term of the turbulent viscosity corresponds to a finite-size effect correction. Note that for a Kolmogorov spectrum, $E(k_c, t)$ is the real energy spectrum of the simulation at time $t$ and $k_m$ is the largest wave number of the simulation (which in our case is $k_m = 41$). The second term of the turbulent viscosity corresponds to a finite-size effect correction.

For our RDT simulation we choose the cut-off wave number $k_c = 11$ in the inertial range so that we can use the “spectral turbulent viscosity model,” which is derived for high Reynolds number. The correlation coefficient between DNS and each model is computed by Eq. (29). Figure 22 shows the correlation coefficient between DNS and RDT without additional correction and between DNS and RDT with an additional turbulent viscosity computed with Eq. (13). Without any correction, the correlation is very good for the first turnover time and then starts to decrease. This effect is due to the fact that the large scales are affected only by the approximation of the small-scale interactions. We have seen that the correction by a turbulent viscosity is usually very efficient in improving the simulation of the intermediate scales.

where $E(q, t)$ is the real energy spectrum of the simulation at time $t$ and $k_m$ is the largest wave number of the simulation (which in our case is $k_m = 41$). The second term of the turbulent viscosity corresponds to a finite-size effect correction. Note that for a Kolmogorov spectrum, Eq. (30) corresponds to Eq. (18) with $C_v = C_v \sqrt{3/20}$. In addition to this turbulent viscosity, a strong hyperviscosity ($10^{-19} \Delta^6$) has been introduced in order to smooth the smallest resolved scales. The constant $C_v$ of the equation (30) is chosen to stabilize the energy spectrum close to a Kolmogorov spectrum (see Fig. 23). There seems to be a weak but discernible dependence of the constant with respect to the cut-off and the resolution. We may interpret this dependence in the conceptual framework of DSTA: Because of scale invariance, the turbulent viscosity exerted at small scales from the smallest scale of the simulation must be equivalent to the turbulent viscosity exerted at the largest scale by all the sub-grid scales up to $k = k_{\text{max}}$. So, we may expect the coefficient to vary as in the DSTA prediction, with $\beta = k_{\text{max}} / k_c$. The comparison between this expectation and the DSTA formula with $\beta = k_{\text{max}} / k_c = 64 / k_c$ is shown in Fig. 24, and indeed there is good quantitative agreement.

D. Comparison with a spectral viscosity model

In order to quantify the improvement of our model, we compare the correlation of the large scales in our RDT model and with the equivalent DNS. This analysis is conducted on forced turbulence over three turnover times. This situation corresponds to a simulation on a $32^3$ grid for the large scales when introducing a $2/3$ rule for de-aliasing. We also compare our RDT model with a simple and practical LES model, the “spectral eddy viscosity model” proposed by Chollet and Lesieur. We choose this model because of its popularity and because it is very easy to implement. A comparison with some models like similarity models or more recent models based on an estimation of the velocity at unresolved scales would be interesting. However, full and detailed comparisons would require a lot more computational effort and are not the main point of this paper. In this model the effects of the sub-grid scales are represented by a turbulent viscosity which is derived from the EDQNM, assuming a $k^{-5/3}$ energy spectra in the inertial range. The turbulent viscosity is linked to the energy $E(k_c, t)$ at the cut-off scales by

\[ \nu_c = 0.267 \frac{\sqrt{E(k_c)}}{k_c}. \]

For our RDT simulation we choose the cut-off wave number $k_c = 11$ in the inertial range so that we can use the “spectral turbulent viscosity model,” which is derived for high Reynolds number. The correlation coefficient between DNS and each model is computed by Eq. (29). Figure 22 shows the correlation coefficient between DNS and RDT without additional correction and between DNS and RDT with an additional turbulent viscosity computed with Eq. (13). Without any correction, the correlation is very good for the first turnover time and then starts to decrease. This effect is due to the fact that the large scales are affected only by the approximation of the small-scale interactions. We have seen that the correction by a turbulent viscosity is usually very efficient in improving the simulation of the intermediate scales. In this case this correction seems to improve the large-scale evolution only slightly. This is probably due to the fact that we chose quite a large cut-off scale and did not try to adapt the definition of the turbulent viscosity to this case. The result is that the intermediate and small scales are much less improved by the correction than in the same simulation with $k_c = 16$. Yet, one can still see a large improvement of our RDT model at large scales compared to the spectral turbulent viscosity model (F-LES) at the same large scale resolution. (The simulation F-LES was conducted on a $32^3$ grid corresponding to 11 Fourier modes in each direction after the de-aliasing procedure.)
plitude (PDFs of velocity differences) at the small scales. Our model also differs from the DIA by the explicit addition of a turbulent viscosity for modeling interactions of inertial-range with near-dissipative-range wave numbers. This feature also appears to be essential in obtaining the correct power-law energy spectrum with the correct magnitude. It is a sensitive parameter of the model since only one value of the viscosity (more specifically, the amount of dissipation at intermediate scales) appears to provide the correct result. We have compared this value with theoretical prescriptions based on recursive analytical approximations of the Navier–Stokes equations. Prescriptions based on RG methods lead to an overestimation of this dissipation. However, a prediction of the DSTA leads to a correct estimate, provided the scale separation parameter is set to the ratio of the integral scale and the cut-off scale.

Our model can also be seen as a sub-grid scale model of turbulence where an approximated linear equation is integrated at small scales in order to calculate the sub-grid–subgrid momentum flux tensor for simulation of the large scales. Since the sub-grid scale sensor seems to be affected mainly by a range of scales that are just smaller than the resolved scales, the main goal is to reproduce these scales as accurately as possible. Our analyses of numerical simulations show that the Langevin RDT model is able to reproduce the correct statistics of both the resolved and the largest sub-grid scales and an accurate evolution of the resolved scales compared to an equivalent DNS. Therefore, this analysis provides a successful validation of the RDT dynamical model of 3D turbulence proposed in this paper. The same model has been used to prescribe analytically the equilibrium mean profiles in channel flow, in the planetary boundary layer, and in thermal convection.

In the case of 2D turbulence, this model leads to significant reduction of computational cost with sufficient accuracy compared to an equivalent DNS. In 3D we expect a similar method to lead to at least the same gain, thereby opening the possibility of simulating numerically flows with much higher Reynolds number than possible with DNS and with greater validity than with conventional LES. This is the subject of ongoing research.

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